

# Profinite completion and double-dual : isomorphisms and counter-examples

Colas Bardavid\*  
IRMAR — UMR 6625 du CNRS  
Université de Rennes 1 Campus de Beaulieu  
35042 Rennes CEDEX FRANCE

---

**Abstract** – We define, for any group  $G$ , finite approximations ; with this tool, we give a new presentation of the profinite completion  $\hat{\pi} : G \rightarrow \hat{G}$  of an abstract group  $G$ . We then prove the following theorem : if  $k$  is a finite prime field and if  $V$  is a  $k$ -vector space, then, there is a natural isomorphism between  $\hat{V}$  (for the underlying additive group structure) and the additive group of the double-dual  $V^{**}$ . This theorem gives counter-examples concerning the iterated profinite completions of a group. These phenomena don't occur in the topological case.

---

## (1) Introduction.

In this paper, we study the profinite completion of a certain class of groups, namely, the additive groups of vector spaces over  $\mathbb{F}_p$ . The principal result is that, in this case, the profinite completion equals the double-dual. This study is based on a “dual” definition of the profinite completion of a group.

---

## (2) Brief survey of the classical point of view for profinite completion.

As explained in [Ser02] or [RZ00], one usually defines the profinite completion of a group<sup>1</sup>  $G$  as follows. The profinite completion  $\hat{G}$  of  $G$  is the projective limit (ie the inverse limit) of the finite quotients of  $G$  :

$$\hat{G} = \varprojlim_{\substack{N \triangleleft G \\ [G:N] < \infty}} G/N.$$

There is a more explicit form for this definition. Indeed, if  $N, M$  are two normal subgroups of  $G$  with  $N \subset M$ , we have a natural factorisation  $\varphi_{N \subset M}$  of the canonical projection  $\pi_M$  :

$$\begin{array}{ccc} & & G/N \\ & \nearrow \pi_N & \downarrow \varphi_{N \subset M} \\ G & & G/M \\ & \searrow \pi_M & \end{array}$$

---

\*colas.bardavid@univ-rennes1.fr

<sup>1</sup>If we set in the category of topological groups, we should precise : “... of a discrete group  $G$ ”.

One can then write :

$$\widehat{G} = \left\{ (x_N) \in \prod_{\substack{N \triangleleft G \\ [G:N] < \infty}} G/N \mid \forall N \subset M, \quad \varphi_{N \subset M}(x_N) = x_M \right\}.$$


---

### (3) Finite approximations and profinite completion.

In this paper, we will use a “dual” (but equivalent) point of view for the profinite completion of a group. To begin with, we introduce the notion of “finite approximation”, which will lead naturally to the concept of profinite completion.

**(3.1) Definition.** If  $G$  is a group, we call finite approximation of  $G$  every couple  $\mathbf{v} = (F, \varphi)$  where  $F$  is a finite group and  $\varphi : G \rightarrow F$  a morphism. We denote  $F = F_{\mathbf{v}}$  and  $\varphi = \varphi_{\mathbf{v}}$ . We say that  $f : \mathbf{v} \rightarrow \mathbf{v}'$  is a morphism between  $\mathbf{v}$  and  $\mathbf{v}'$  if it is an arrow that makes the following diagram commute :

$$\begin{array}{ccc} & & F_{\mathbf{v}} \\ & \nearrow \varphi_{\mathbf{v}} & \\ G & & \\ & \searrow \varphi_{\mathbf{v}'} & \\ & & F_{\mathbf{v}'} \end{array} \quad \begin{array}{c} \\ \\ f \\ \\ \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \end{array}$$

We denote  $\mathbf{App}_f(G)$  the category of finite approximations of  $G$ .

Intuitively, a finite approximation of  $G$  allows the mathematician to get some information about  $G$  by only dealing with finite objects. Here are some examples, from various aeras of mathematics, of finite approximations :

- a)  $\mathbf{R}^* \longrightarrow \mathbf{Z}/2\mathbf{Z}$   
 $x \longmapsto \text{sgn}(x)$  the sign of a real number.
- b) The reduction modulo  $n$ ,  $\mathbf{Z} \rightarrow \mathbf{Z}/n\mathbf{Z}$  and all the derived morphisms and generalizations, such that  $\mathbf{Z}_{(p)} \rightarrow \mathbf{Z}/p\mathbf{Z}$ , such that  $GL_m(\mathbf{Z}) \rightarrow GL_m(\mathbf{Z}/n\mathbf{Z})$  or such that  $\mathcal{O}_K \rightarrow \mathcal{O}_K/\mathfrak{P}$  if  $K$  is a number field ;
- c) If  $X$  a topological space with a finite number of connected components, we can consider the “trace” on  $\pi_0(X)$  of an automorphism :  $\begin{array}{ccc} \text{Aut}(X) & \longrightarrow & \mathfrak{S}_{\pi_0(X)} \\ \phi & \longmapsto & \pi_0(\phi) \end{array}$ .
- d) If we denote  $\mathfrak{S}_{(\mathbf{N})} = \varinjlim_n \mathfrak{S}_n$  the group of permutation of  $\mathbf{N}$  with finite support, we can still define a signature  $\mathfrak{S}_{(\mathbf{N})} \rightarrow \mathbf{Z}/2\mathbf{Z}$ .
- e) Finally, if  $K/\mathbf{Q}$  is a Galois extension, then  $\begin{array}{ccc} \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) & \longrightarrow & \text{Gal}(K/\mathbf{Q}) \\ \sigma & \longmapsto & \sigma|_K \end{array}$  is a finite approximation.

**(3.2) Profinite completion.** Then, one can define very naturally the profinite completion of  $G$  as the projective limit of all the finite approximations of  $G$ . More precisely, (and without dealing with any problem of set theory)

$$\widehat{G} = \left\{ (g_v)_{v \in \mathbf{App}_f(G)} \in \prod_v F_v \mid \forall \psi : v \rightarrow w, \psi(g_v) = g_w \right\}$$

which comes with the *profinite projection*

$$\widehat{\pi} : \begin{array}{c} G \longrightarrow \widehat{G} \\ g \longmapsto (\varphi_v(g))_{v \in \mathbf{App}_f(G)} \end{array} .$$

Intuitively, this object is what remains from  $G$  when one can only deal with information of finite type ; some elements will be identified but, at the same time, some new elements will appear. Formally, in general,  $\widehat{\pi}$  is not surjective or injective.

**(3.3) Surjective finite approximations.** Among the finite approximations, some are surjective ; they form a full subcategory  $\mathbf{App}_f^s(G)$  of  $\mathbf{App}_f(G)$ . In the same way that we have defined the profinite completion, we can then define the “surjective” profinite completion

$$\widehat{G}^s = \varprojlim_{v \in \mathbf{App}_f^s(G)} F_v .$$

The important fact about this object is that we have the following fact, whose proof is not difficult.

**(3.4) Proposition.** *The natural morphism  $\widehat{G} \rightarrow \widehat{G}^s$  is an isomorphism.*

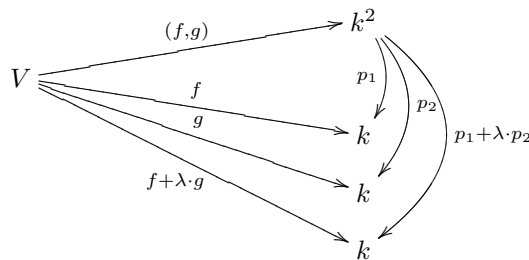
#### (4) Profinite completion of the additive group of a vector space over $\mathbf{F}_p$ .

**(4.1) Profinite completion and double-dual.** Before looking at what happens in the situation where the base field is  $\mathbf{F}_p$ , let us remark that, in the general case, there is a morphism of comparison between the profinite completion of an “additive group” and its double-dual. Let  $k$  be a finite field and  $V$  a vector space over  $k$ . We still denote by  $V$  the underlying additive group.

Let  $f$  and  $g$  be two linear forms of  $V$  and let  $\lambda \in k$ . The forms  $f$ ,  $g$  and  $f + \lambda \cdot g$  are, in particular, finite approximations of  $V$  (in the additive group of  $k$ ) and we denote by  $v_f$ ,  $v_g$  and  $v_{f+\lambda \cdot g}$  the corresponding approximations. Now, let  $x = (x_v)_v \in \widehat{V}$  be a “profinite” element.

**(4.2) Fact.**  $x_{v_{f+\lambda \cdot g}} = x_{v_f} + \lambda \cdot x_{v_g}$ .

**Proof :** Indeed, we have the following diagram of morphisms of finite approximations



Then, if we denote by  $\mathbf{w}$  the approximation  $V \xrightarrow{(f,g)} k^2$ , the definition of the profinite completion imposes that  $x_{\mathbf{v}_f} = p_1(x_{\mathbf{w}})$  and  $x_{\mathbf{v}_g} = p_2(x_{\mathbf{w}})$  and  $x_{\mathbf{v}_{f+\lambda \cdot g}} = (p_1 + \lambda \cdot p_2)(x_{\mathbf{w}})$ , that is

$$x_{\mathbf{v}_{f+\lambda \cdot g}} = x_{\mathbf{v}_f} + \lambda \cdot x_{\mathbf{v}_g}.$$

■

Using this fact, one can define the morphism of comparison :

$$\begin{aligned} \Psi : \quad \widehat{V} &\longrightarrow V^{**} \\ (x_{\mathbf{v}})_{\mathbf{v}} &\longmapsto \left( \begin{array}{c} V^* \longrightarrow k \\ f \longmapsto x_{\mathbf{v}_f} \end{array} \right) \end{aligned}$$

**(4.3) The case where  $k = \mathbf{F}_p$ .** From now on,  $p$  is a prime number and  $k = \mathbf{F}_p$ . The interesting case is when  $V$  is of infinite dimension. A good way to understand what happens is to consider  $V = (\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$ .

The first thing to do is to see that if  $\varphi : V \rightarrow F$  is a finite *surjective* approximation, then  $F$  is isomorphic to (the additive group of)  $(\mathbf{F}_p)^n$  for some  $n$ . Indeed, first of all, since  $F$  is the homomorphic image of  $V$ ,  $F$  is abelian. Moreover, all the elements of  $F$  satisfy  $x^p = e$ . Thus, the classification of the abelian finite groups gives the conclusion.

We can now prove :

**(4.4) Theorem** *Let  $V$  be a vector space over  $\mathbf{F}_p$ . Then,  $\Psi : \widehat{V} \rightarrow V^{**}$  is an isomorphism.*

**Proof :** We first prove that  $\Psi$  is injective : let  $\mathbf{x} = (x_{\mathbf{v}})_{\mathbf{v}} \in \widehat{V}$  such that for all linear form  $f : V \rightarrow k$ ,  $x_{\mathbf{v}_f} = 0$ . Let  $\mathbf{v}$  be a finite surjective approximation of  $V$  ; we can suppose that  $\mathbf{v} = (k^n, \varphi)$ , where  $\varphi : V \rightarrow k^n$  is any morphism. By composing  $\varphi$  with the  $n$  projections  $p_i$  to the factors  $k$ , one obtain  $n$  morphisms. If we prove that the  $n$  corresponding elements are equal to 0, then, it will follow that  $x_{\mathbf{v}}$  is equal to 0 and, thus, that  $\Psi$  is injective. But, and it is the (easy) key point, a morphism  $V \rightarrow k$  of groups is actually a linear form, since we can rewrite the condition  $\varphi(\lambda \cdot \vec{v}) = \lambda \cdot \varphi(\vec{v})$  as  $\varphi(\vec{v} + \dots + \vec{v}) = \varphi(\vec{v}) + \dots + \varphi(\vec{v})$ , for our base field is  $\mathbf{F}_p$ . And, by assumption, all the  $x_{\mathbf{v}_f} = 0$ .

For the surjectivity, let  $\Theta \in V^{**}$  be a double-dual element. We would like to find a profinite element  $\mathbf{x} = (x_{\mathbf{v}})_{\mathbf{v}} \in \widehat{V}$  such that, for all linear form  $f$  of  $V$ , one have  $x_{\mathbf{v}_f} = \Theta(f)$ . So, let  $\mathbf{v} = (k^n, \varphi)$  (as we can suppose it) be a finite approximation of  $V$ . Let denote  $p_1, \dots, p_n$  the  $n$  projections of  $k^n$  to the factors  $k$ . Naturally, we define  $x_{\mathbf{v}}$  by reconstructing it from the linear forms  $p_i \circ \varphi$  :

$$x_{\mathbf{v}} := (\Theta(p_1 \circ \varphi), \Theta(p_2 \circ \varphi), \dots, \Theta(p_n \circ \varphi)) \in k^n.$$

Now, we just have to check that the family  $(x_{\mathbf{v}})$  is “compatible“. So let  $\mathbf{v} = (k^n, \varphi)$  and  $\mathbf{w} = (k^m, \psi)$  be two finite approximations and  $g$  a morphism between them :

$$\begin{array}{ccc} & & k^n \\ & \nearrow \varphi & \downarrow g \\ V & & k^m \\ & \searrow \psi & \end{array}$$

We want to prove that  $g(x_{\mathbf{v}}) = x_{\mathbf{w}}$ . By composing with the  $m$  projections  $q_i$  of  $k^m$ , it suffices to prove it in the case where  $m = 1$  :

$$\begin{array}{ccccc} & & k^n & & \\ & \nearrow \varphi & \downarrow g & & \\ V & & k^m & \xrightarrow{q_1} & k \\ & \searrow \psi & & \searrow q_m & \\ & & & & \vdots \\ & & & & k \end{array}$$

So, we are brought to this situation

$$\begin{array}{ccc} & & k^n \\ & \nearrow \varphi & \downarrow g \\ V & & k \\ & \searrow \psi & \end{array}$$

where we know that  $g$  can be written as  $g = \lambda_1 \cdot p_1 + \dots + \lambda_n \cdot p_n$ , with  $\lambda_i \in k$ . The fact that the previous diagram commutes tells us that  $\psi = \sum_i \lambda_i \cdot (p_i \circ \varphi)$ ; and, now :

$$\begin{aligned} g(x_{\mathbf{v}}) &= g((\Theta(p_1 \circ \varphi), \Theta(p_2 \circ \varphi), \dots, \Theta(p_n \circ \varphi))) \\ &= \sum_i \lambda_i \cdot \Theta(p_i \circ \varphi) \\ &= \Theta\left(\sum_i \lambda_i \cdot (p_i \circ \varphi)\right) = \Theta(\psi) \\ &= x_{\mathbf{w}}, \end{aligned}$$

which concludes the proof. ■

**(4.5)  $\hat{\pi}$  and the canonical injection  $i : V \rightarrow V^{**}$ .** We denote  $i : V \rightarrow V^{**}$  the canonical injection defined by  $i(\vec{v})(f) = f(\vec{v})$ . One can improve a bit the theorem 4.4 : the isomorphism  $\Psi$  between  $\hat{V}$  and  $V^{**}$  through  $\Psi$  identifies  $\hat{\pi}$  with  $i$ . The proof is easy.

**(4.6) Theorem.** *Let  $V$  be a vector space over  $\mathbf{F}_p$ . Then,  $\Psi : \hat{V} \rightarrow V^{**}$  is an isomorphism and the diagram*

$$\begin{array}{ccc} & & \hat{V} \\ & \nearrow \hat{\pi} & \downarrow \Psi \\ V & & V^{**} \\ & \searrow i & \end{array}$$

*commutes.*

**(4.7) Remark.** One can prove the theorem 4.4 with more abstracted arguments. To begin with, we know (cf. for example [Par70, §2.7, theorem 2]) that, in a general category  $\mathcal{C}$ , if the limits exist, we always have the natural isomorphism

$$\text{Hom}\left(\lim_{\longrightarrow i} X_i, X\right) \cong \lim_{\longleftarrow i} \text{Hom}(X_i, X).$$

Moreover, in the case of  $k$ -vector spaces, this isomorphism is linear ; thus, for a system of  $k$ -vector space  $V_i$ , we have :

$$\left(\lim_{\longrightarrow i} V_i\right)^* \cong \lim_{\longleftarrow i} (V_i^*).$$

Let  $k$ , from now on, be a field and  $V$  a  $k$ -vector space. If we denote by  $(Y_i)_i$  the system of finite-dimensional subvector spaces of  $V^*$ , we have  $V^* = \lim_{\longrightarrow i} Y_i$  and thus, thanks the previous isomorphism :

$$V^{**} \cong \lim_{\longleftarrow i} (Y_i^*).$$

Moreover, there is a natural bijection between the finite-dimensional subspaces of  $V^*$  and the finite-codimensional subspaces of  $V$ , via the application

$$Y \mapsto Y^\perp := \{v \in V \mid \forall \varphi \in Y, \varphi(v) = 0\}.$$

Besides, if  $Y$  is a finite-dimensional subspace of  $V^*$  then the dual  $Y^*$  is naturally isomorphic to  $V/Y^\perp$ . Consequently, if we denote by  $(Z_j)_j$  the system of finite-codimensional subspaces of  $V$ , we have :

$$V^{**} \cong \varprojlim_j (V/Z_j).$$

But, if  $k = \mathbf{F}_p$  for a prime number  $p$ , one can identify the  $k$ -vector space  $V$  with its underlying additive group<sup>2</sup>  $\omega(V)$ , its dual  $V^*$  with  $\text{Hom}_{\mathbf{Gr}}(\omega(V), \omega(\mathbf{F}_p))$ , and its finite dimensional quotients with the finite quotient of  $\omega(V)$ . We thus finally get the expected alternative proof of the theorem 4.4.

---

### (5) A family of counter-examples.

One would like to know if, given a group  $G$ , one have  $\widehat{\widehat{G}} \simeq \widehat{G}$ . This fact is known to be false (cf. example 4.2.13 of [RZ00]), but as we will see, it is still false, in general, after taking  $i$  times the profinite completion.

**(5.1) The sequence of  $i$ -th profinite completions.** We introduce the following notation. If  $G$  is a group, we denote  $\widehat{G}^{[1]} = \widehat{G}$  and  $\widehat{G}^{[i+1]} = \widehat{\widehat{G}^{[i]}}$ . These groups come with projections, as follows :

$$G \xrightarrow{\widehat{\pi}^{[1]}} \widehat{G}^{[1]} \xrightarrow{\widehat{\pi}^{[2]}} \widehat{G}^{[2]} \longrightarrow \cdots \longrightarrow \widehat{G}^{[i]} \xrightarrow{\widehat{\pi}^{[i+1]}} \widehat{G}^{[i+1]} \longrightarrow \cdots.$$

We will prove that, in general, none of the  $\widehat{\pi}^{[i]}$  is an isomorphism.

**(5.2) Proposition.** *Let  $p$  be a prime number and  $k = \mathbf{F}_p$ . Let  $V$  be (the additive group of) a  $k$ -vector space of infinite dimension. Then, in the following sequence*

$$V \xrightarrow{\widehat{\pi}^{[1]}} \widehat{V}^{[1]} \xrightarrow{\widehat{\pi}^{[2]}} \widehat{V}^{[2]} \longrightarrow \cdots \longrightarrow \widehat{V}^{[i]} \xrightarrow{\widehat{\pi}^{[i+1]}} \widehat{V}^{[i+1]} \longrightarrow \cdots$$

*all the  $\widehat{\pi}^{[i]}$  are injective but non-surjective morphisms.*

**Proof :** This follows from the identification of the arrows  $\widehat{\pi}^{[i]}$  with the canonical injections of a vector space in its double-dual, and from the fact that these injections are injective but non-surjective when the vector spaces are of infinite dimension, cf. Théorème 6, §7, n°5 of [Bou62]. ■

---

<sup>2</sup>We denote  $\omega : k\text{-}\mathbf{Vs} \rightarrow \mathbf{Gr}$  the forgetful functor from the category of  $k$ -vector spaces to the category of groups.

## (6) Conclusion : abstract setting vs. topological setting.

This study has been given for groups but a similar point of view can be applied to *topological* groups. In this case, we start with a topological group  $\mathcal{G}$  and we consider the category  $\mathbf{App}_{discr}(\mathcal{G})$  of finite and *discrete* approximations : they are couples  $\mathbf{v} = (F, \varphi)$ , where  $F$  is a discrete and finite topological group and  $\varphi : \mathcal{G} \rightarrow F$  a continuous morphism of groups.

One obtain the (topological) profinite completion of  $\mathcal{G}$ , wich is, as well-known, a topological group, compact and totally disconnected (cf. [Ser02]), and one obtain a profinite projection, which is a continuous morphism :

$$\widehat{\pi}^{top} : \mathcal{G} \rightarrow \widehat{\mathcal{G}}^{top}.$$

More generally, as previously done, one can define the sequence of iterated (topological) profinite completions :

$$\mathcal{G} \xrightarrow{\widehat{\pi}^{[1],top}} \widehat{\mathcal{G}}^{[1],top} \longrightarrow \dots \longrightarrow \widehat{\mathcal{G}}^{[i],top} \xrightarrow{\widehat{\pi}^{[i+1],top}} \widehat{\mathcal{G}}^{[i+1],top} \longrightarrow \dots.$$

The situation is then totally different than before. Indeed, we have :

**(6.1) Proposition.** *Let  $\mathcal{G}$  be a topological group. Then, for all  $i \geq 2$ , the arrows  $\widehat{\pi}^{[i],top}$  are isomorphisms of topological groups.*

**(6.2) Profinite groups : abstract setting and topological setting.** There is a synthetical way to see the fundamental difference between the propositions 4.2 and 6.1. For this sake, we introduce two notions of profinite groups. We will say that a group  $G$  is *profinite* if it is the projective limit of a system of finite groups ; we will say that a topological group  $\mathcal{G}$  is *topologically profinite* if it is the projective limit of a system of finite and discrete groups. We then have :

**(6.3) Theorem** *Let  $\mathcal{G}$  be a topological group. Then :*

$$\mathcal{G} \text{ is topologically profinite} \iff \widehat{\pi}^{top} : \mathcal{G} \rightarrow \widehat{\mathcal{G}}^{top} \text{ is an isomorphism.}$$

**(6.4) Proposition** *Let  $G$  be a group. Then :*

$$G \text{ is profinite} \iff \widehat{\pi} : G \rightarrow \widehat{G} \text{ is an isomorphism}$$

$$G \text{ is profinite} \nRightarrow \widehat{\pi} : G \rightarrow \widehat{G} \text{ is an isomorphism.}$$

**(6.5) A positive answer.** One could legitimately be disapointed by the non-equivalence of  $G$  being profinite and of  $\widehat{\pi}$  being an isomorphism. Indeed, on the one hand, there is the very classical definition of a profinite group and, on the other hand, there is the deep property for a group to have its profinite projection  $\widehat{\pi}$  to be an isomorphism (such a group, in a way, is separated — for  $\widehat{\pi}$  is injective — and complete — for  $\widehat{\pi}$  is surjective). One would have expected these two to coincide...

Fortunately, there is a positive result in this direction. It is a difficult result, which has been published in 2007 by Nikolay Nikolov and Dan Segal, cf. [NS07a] and [NS07b], and whose proof uses the classification of finite simple groups. In order to state their result, let us remark that if  $G$  is an (abstract) profinite group, if we write  $G = \varprojlim_i F_i$ , where the  $F_i$ 's are finite, and if we endow each of the  $F_i$ 's with the discrete topology, then we can view  $G$  as a topological group.

**(6.6) Theorem** *Let  $G$  be an (abstract) profinite group, which is topologically of finite type for the associated topology. Then,  $\widehat{\pi} : G \rightarrow \widehat{G}$  is an isomorphism.*

---

**(6.7) Acknowledgments.** My first acknowledgments go to Xavier Caruso for many helpful discussions. I would like also to thank the referee for many valuable comments and for making me known the alternative proof 4.7.

## References

- [Bou62] Nicolas Bourbaki. *Éléments de mathématique. Première partie. Fascicule VI. Livre II: Algèbre. Chapitre 2: Algèbre linéaire.* Troisième édition, entièrement refondue. Actualités Sci. Indust., No. 1236. Hermann, Paris, 1962.
- [NS07a] Nikolay Nikolov and Dan Segal. On finitely generated profinite groups. I. Strong completeness and uniform bounds. *Ann. of Math. (2)*, 165(1):171–238, 2007.
- [NS07b] Nikolay Nikolov and Dan Segal. On finitely generated profinite groups. II. Products in quasisimple groups. *Ann. of Math. (2)*, 165(1):239–273, 2007.
- [Par70] Bodo Pareigis. *Categories and functors.* Translated from the German. Pure and Applied Mathematics, Vol. 39. Academic Press, New York, 1970.
- [RZ00] Luis Ribes and Pavel Zalesskii. *Profinite groups*, volume 40 of *Ergeb. Math. Grenzgeb. (3)*. Springer-Verlag, Berlin, 2000.
- [Ser02] Jean-Pierre Serre. *Galois cohomology.* Springer Monogr. Math. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.